B. Math. III – Mid-Term Examination

Introduction to Differential Geometry

September 23, 2008

1. Let S be an open subset of $\mathbb{R}^n$, and let $f : S \to \mathbb{R}$ be a real-valued function with finite partial derivatives $D_1 f, \ldots, D_n f$ on S. If f has a local maximum or a local minimum at a point c in S, then prove that $D_k f(c) = 0$ for each k.

2. Let γ be a curve in $\mathbb{R}^n$ and let $\widetilde{\gamma}$ be its reparametrisation with the reparametrisation map ϕ (hence $\widetilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$. Let s and $\tilde{s}$ be arc lengths of γ and $\widetilde{\gamma}$ starting at a point $P = \gamma(t_0) = \widetilde{\gamma}(\tilde{t}_0)$. Prove that $\tilde{s} = \pm\, s$, with the sign being determined by the monotonicity of ϕ.

3. Define the vector product of two vectors in $\mathbb{R}^3$. Prove that the norm of the vector product of the two vectors u and v is the area of the parallelogram generated by them.

4. Let γ be a unit-speed plane curve with unit signed normal vector n_s and signed curvature κ_s. Let λ be a constant. Define the *parallel curve* γ^λ of γ by:

$$\gamma^\lambda(t) = \gamma(t) + \lambda n_s(t).$$

Show that γ^λ is regular if $|\lambda \kappa_s(t)| < 1$ for all values of t. Compute the signed curvature of γ^λ.

5. Show that the ellipse

$$\gamma(t) = (a\cos(t), b\sin(t)),$$

where a and b are positive constants, is a simple closed curve and compute the area of its interior.

6. Let γ be a unit speed curve in $\mathbb{R}^3$ and σ be the ruled surface

$$\sigma(u, v) = \gamma(u) + v\delta(u),$$

where $\delta(u)$ is a unit vector for all u. Prove that σ is conformal (patch) if and only if δ is a constant function of u and γ is a curve in a plane that is orthogonal to δ.